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Dynamics towards synchronization in hierarchical networks

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Abstract

We review some of the results obtained in our group in recent years concerning the dynamical behavior of phase oscillators in networks with a hierarchical structure. The dynamics highlights the role of communities at all levels. At the same time, by analyzing the resulting dynamical evolution we can infer details about the topological structure of the system. Thus a link between dynamical and topological properties of complex systems is established.

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1. Introduction

The theory of complex networks has reported major advances in the understanding of the substrate in which many natural, social and technological processes take place. Complex networks are representative of the intricate connections between elements in systems as diverse as the Internet and the WWW, metabolic networks, neural networks, food webs, communication networks, transport networks and social networks. The availability of wide databases of entities (nodes) and relations (links) as well as the advances in computation have provided scientists with the necessary tools to unravel the statistical properties of complex networks [1–3].

One of the subjects that has received more attention, in recent years, is the detection and characterization of intermediate topological scales in their structure. In particular, the problem of the detection of *community structure*, meaning the appearance of densely connected groups of vertices, with only sparser connections between groups, has been intensely attacked from the scientific community [4, 5].

In a different scenario, physicists have largely studied the dynamics of complex biological systems, and in particular the paradigmatic analysis of large populations of coupled oscillators [6–8]. The connection between the study of synchronization processes and complex networks is interesting in itself. These synchronization phenomena have been mainly described under the mean field hypothesis that assumes that all oscillators behave identically and interact with the rest of the population. Recently, the emergence of synchronization phenomena in these systems has been shown to be closely related to the underlying topology of interactions at mesoscopic scales [9].

Here we review the effect of the community structure in the path towards the synchronization of a population of identical oscillators. We study the dynamical behavior in several types of structured complex networks and find an evolving community structure based on the recruitment of groups of nodes towards complete synchronization. The paper is structured as follows: in section 2 we present the synchronization model studied. In section 3 we describe the synthetic networks with a well-prescribed hierarchical community structure. In section 4, we expose the analysis of the route towards synchronization and their relationship with the topological structure. Finally, we end with the main conclusions extracted from the presented approach.

2. Dynamical model

One of the most successful attempts to understand synchronization phenomena in physics was due to Kuramoto [7], who proposed a model of oscillators in which their phases are coupled in the form of a sine function of their differences. The model is rich enough to display a large variety of synchronization patterns and sufficiently flexible to be adapted to many different contexts [10]. The model we propose here consists of a population of N coupled phase oscillators where the phase of the ith unit, denoted by $\theta_i(t)$, evolves in time according to the following dynamics:

$$\frac{\mathrm{d}\theta_i}{\mathrm{d}t} = \omega_i + \sum_j K_{ij} \sin(\theta_j - \theta_i) \qquad i = 1, \ldots, N \tag{1}$$

where ω_i stands for its natural frequency and K_{ij} describes the coupling between units. The original model studied by Kuramoto assumed mean-field interactions $K_{ij} = K, \forall i, j$. In the absence of noise the long time properties of the population are determined by analyzing the only two factors which play a role in the dynamics: the strength of the coupling K whose effect tends to synchronize the oscillators (same phase) versus the width of the distribution of natural frequencies, the source of disorder which drives them to stay away from each other by running at different velocities.

Recently, due to the realization that many networks in nature have complex topologies, these studies have been extended to systems where the pattern of connections is local but non-trivial [11–18]. Usually, due to the complexity of the analysis some further assumptions have been introduced. For instance, it has been a normal practice to assume that the oscillators are identical. If ($\omega_i = \omega \forall i$) there is only one attractor of the dynamics: the fully synchronized regime where $\theta_i = \theta, \forall i$. In this context the interest concerns the route to the attractor. In particular, it has been shown [19, 20] that high densely interconnected sets of oscillators (motifs) synchronize more easily that those with sparse connections. This scenario suggests that for a complex network with a non-trivial connectivity pattern, starting from random initial conditions, those highly interconnected units forming local clusters will synchronize first and then, in a sequential process, larger and larger spatial structures also will do it up to the final state where the whole population should have the same phase. This process occurs in a progressive way at different time scales if a clear community structure exists. Thus, the dynamical route towards the global attractor reveals different topological structures, presumably those which represent communities. Therefore, it is the complete dynamical process what unveils the whole organization at all scales, from the microscale at a very early stages up to the macroscale at the end of the time evolution. In contrast, those systems endowed with a regular topological structure will display a trivial dynamics with a single time scale for synchronization.

It is a normal practice to define, for the Kuramoto and related models, a global 'order parameter' to characterize the level of entrainment between oscillators. However, this definition, although suitable for mean-field models, is not efficient to identify local dynamic effects. In particular, it does not give information about the route to the attractor (fully synchronization) in terms of local clusters which is so important to identify functional groups or communities. For this reason, instead of considering a global observable, we define a local order parameter measuring the average of the correlation between pairs of oscillators [21]:

$$\rho_{ij}(t) = \langle \cos(\theta_i(t) - \theta_j(t)) \rangle \tag{2}$$

where the brackets stand for the average over initial random phases. The main advantage of this approach is that it allows us to trace the time evolution of pairs of oscillators and therefore to identify compact clusters of synchronized oscillators reminiscent of the existence of communities.

3. Hierarchical networks

To give evidence of the aforementioned facts we have analyzed the dynamics towards synchronization—time evolution of $\rho_{ij}(t)$—in computer-generated networks with a clear community structure. Some of them are homogeneous in degree, whereas other networks have special nodes that act as hubs.

The networks we have analyzed are as follows:

- *Networks with one level of homogeneous communities.* It is a kind of network that has been proposed as a benchmark for community detection algorithms, it consists of 128 nodes forming four groups of 32 nodes each. By keeping a fixed number of 16 connections for each node there is a variable number of internal and external connections allowing then a variable degree of compactness of the groups. [5, 22].
- *Networks with two and three hierarchical levels of homogeneous communities.* This generalization was proposed [21, 23] to show that the synchronization dynamics is able to find communities at different levels.
  In general, to construct such a network, one takes a set of N nodes and divides it into n_1 groups of equal size; each of these groups is then divided into n_2 groups and so on up to a number of steps k which defines the number of hierarchical levels of the network. Then we add links to the networks in such a way that at each node we assign at random a number of z_1 neighbors within its group at the first level, z_2 neighbors within the group at the second level and so on. There is a remaining number of links that each node has to the rest of the network, which we will call z_{out}.
  We have considered networks with two hierarchical levels, 256 nodes, and $n_1 = n_2 = 4$; this gives two possible partitions: one with 4 communities and the other one with 16 communities. In the case of three levels we have taken 64 nodes and $n_1 = n_2 = n_3 = 2$, and hence there are three possible partitions, 2, 4 and 8 equal size communities.
- *Hierarchical networks with hubs.* There is a set of self-similar deterministic networks that has been used as an example of hierarchical scale-free networks, proposed by Ravasz and Barabasi [24]. This type of networks, apart from its hierarchical structure has some nodes with a special role in terms of number of connections (hubs) in contrast to the networks discussed previously that are essentially homogeneous in degree.
- *Including communities of different sizes.* In [25] it is shown that the algorithms for detecting the community structure are very sensitive to the size of the communities

themselves, and a model to construct networks with inhomogeneous distributions of communities is proposed. In this case the networks are parameterized by two quantities, the internal and the external cohesion.

4. Dynamics and topology

If we let the system start from random initial conditions and follow its dynamical evolution it will evolve towards a synchronized state, the only attractor for the dynamics, but in a way that will depend strongly on the initial values. If close oscillators have similar phases they will become synchronized quite fast but if, by chance, the phases are far then synchronization will appear later. In order to avoid such a spurious dependence we average over initial conditions. In this way we expect that closer oscillators will synchronize faster and that dynamics towards the final synchronized state will depend only on the topology and not on the particular choice of the initial phases.

The correlation matrix (2) hence contains all the dynamical information of the route towards the final synchronized state. Any element of this matrix is a monotonously increasing function of time. In any case, due to the continuous nature of the phase the completely synchronized state is never reached and the introduction of some sort of threshold above which two oscillators can be considered as synchronized is necessary. For instance, one can measure the time for two oscillators to get entrained [23] or the time needed by the whole system to get completely synchronized [26].

Under these assumptions the visualization of the dynamic evolution of the correlation matrix of the system helps in elucidating the topology of the network. To extract the quantitative information we introduce a threshold T to convert the correlation matrix into a binary matrix which will be used to determine the borders between different groups. We define then a *dynamic connectivity* matrix

$$\mathcal{D}_t(T)_{ij} = \begin{cases} 1 & \text{if} \quad \rho_{ij}(t) > T \\ 0 & \text{if} \quad \rho_{ij}(t) < T \end{cases} \tag{3}$$

that depends on both the underlying topology and the collective dynamics. For a fixed time t, by moving the threshold T, we obtain different representations of $\mathcal{D}_t(T)$ that inform about the structure of the dynamic correlations. When the threshold is large enough the representation of $\mathcal{D}_t(T)$ becomes a set of disconnected clumps or communities. Decreasing T a hierarchical structure of communities is devised. Note that since the function $\rho_{ij}(t)$ is continuous and monotonic (because of the existence of a unique attractor of the dynamics), we can redefine $\mathcal{D}_T(t)$, i.e., fixing the threshold and evolving in time. We obtain the same information about the structure of the dynamic connectivity matrix at different time scales. These time scales unravel the topological structure of the connectivity matrix at different topological scales [21].

In our previous works [21, 23] we presented the results of this procedure by means of several graphical visualizations.[1] One of them is the number of connected components of the dynamic connectivity matrix as a function of time. We showed that there is a striking similarity between this picture and the one obtained by plotting the eigenvalues of the Laplacian matrix [21]. Actually the community structure is related to the existence of jumps in the spectral representation. Furthermore, in [26] we related one of the eigenvalues, the smallest nonzero eigenvalue, also called the *spectral gap*, with the time needed for the whole system to synchronize.

[1] We refer the interested reader to http://www.ffn.ub.es/albert/synchro.html, where a set of visualizations for each network is displayed.

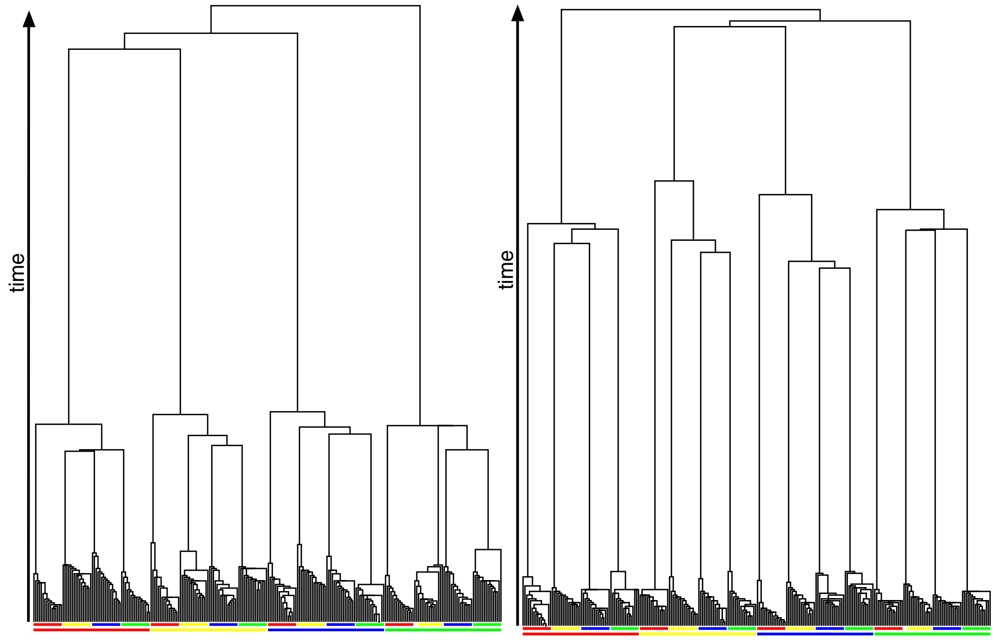

Figure 1. Community merging procedure. The left and right panels correspond to slightly different networks, both of them consisting in two hierarchical levels of communities as described in the text. The color codes on the bottom stand for the hierarchically organized groups. Each pair of oscillators are merged when the value of the corresponding element of the correlation matrix is above some threshold—i.e., they are synchronized. Groups of oscillators already synchronized merge other groups when any of the elements of one of the groups synchronizes with any other element of the other group.

A mode detailed analysis of the dynamical evolution can be obtained by means of the dendogram of the merging of the synchronization clusters, as depicted in figure 1. In this case we consider oscillators to merge if its corresponding element of the dynamic connectivity matrix becomes 1. Here we show as an example of application of this procedure the case of networks with two hierarchical levels of communities that are slightly different at the intermediate scale. We can see that innermost clusters synchronize very fast, whilst the intermediate level synchronizes in a time scale that is very sensitive to the topological structure. In this way we can see how the details of the community structure affect the dynamical evolution of the system. As additional information we can identify the length of the branches of the dendogram as the relative stability of the group. We have carefully analyzed this route to the synchronized state for the whole set of networks described in the previous section. In all of them we can visualize the community structure at all topological levels whenever communities are clearly defined and the networks are basically homogeneous in degree. On the other hand, from the analysis performed on the inhomogeneous networks proposed in [24] we can conclude that hubs play a very special role. Even if they are tightly connected to the rest of the group its special role make them to be synchronized at times later than expected.

An alternative picture of the time evolution is shown in figure 2. In this type of picture we can again identify the two levels of the hierarchical organization of the system. The main advantage of this tool is that it shows strictly the one-to-one relation between the oscillators and it does not imply a transitivity relation as in the dendrogram picture.

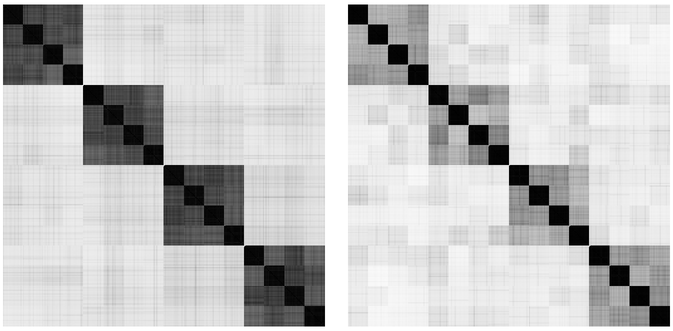

Figure 2. Time needed for each pair of oscillators to synchronize for the same networks as in previous figure. The darker the square the shorter the time for a pair of oscillators to get synchronized.

Finally we would like to emphasize the role of the magnitude that has been widely used in the analysis of networks with community structure, the modularity. Given a network structure the best partition into communities is the one that maximizes the modularity. In [27] we show that meta-stable patterns of synchronization in the path towards complete synchronization are closely related to the partitions obtained optimizing modularity on complex networks [5]. In particular, in networks with homogeneous degree distributions the correspondence is very precise and the larger the modularity of a given partition the more stable the synchronized cluster is. However, in networks where there are hubs, the correspondence fails. Hubs take longer to synchronize and hence they tend to be isolated in our synchronization route. This fact makes a big difference in the way synchronization evolves and the way modularity is heuristically optimized. Of course, modularity optimization is not a dynamical process and hence it does not take into account the relative stability of patterns. If two configurations are equivalent, in the sense that the modularity gain is the same at some point in the optimization process then one chooses at random one of the two configurations. This is what happens for instance when considering isolated nodes, since there will never be isolated nodes in an optimal modularity partition. But from a dynamical point of view such a node, or group of nodes, which has to decide in joining one larger group or another can stay longer in this isolation because it is dynamically stable. Finally it will join one larger group or the other but it can be along a slow process in which the merging into the group is followed by a subsequent merging of the full group of nodes.

We understand what happens in a particular model of phase oscillators, but of course this type of behavior can be quite dependent on the model implementation. In [26] we analyzed a few examples of dynamics. A sine coupling dynamics can be well approximated by its linearized version and hence the time evolution is mainly governed by the Laplacian matrix. On the other hand, a system of spins is shown to evolve according to a matrix that is closely related to the adjacency matrix. We have, then, a couple of cases in which the evolution is governed by different rules and hence we should not expect that a description based solely on topology, as the modularity is, can explain completely the observed behavior.

5. Conclusions

We have presented some results concerning the route to synchronization of networks of phase oscillators. There is clear evidence that the internal structure, in terms of not only communities but also of hubs, is very important in the dynamical evolution of the system. On the other hand, by looking at this dynamical evolution with the appropriate tools we can infer some of the relevant topological features of the system. Finally, the spectral properties of the network, in terms of the eigenvalues of the Laplacian matrix, turn out to be clearly correlated as well with the observed dynamical patterns. This paper emphasizes the interrelation between topology, dynamics and spectral properties of complex topologies. Although many of the times this relation is proven to be fruitful, it has to be used with care. A measure of the accuracy of a network partition into static communities as given by the modularity is not more appropriate to explain the dynamical process of merging of the oscillators into synchronization clusters. It cannot explain the role of special nodes or groups of nodes in the dynamical process. Furthermore, it is possible that other dynamics, described by other rules, need other representations.

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